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ON THE TWO-MODE LASER MASTER EQUATION

Fam Le Kien, A.S.Shumovsky

We use the two-mode laser master equation previously derived by us to investigate the photon dynamics and statistics of the laser. Closed sets of equations for the mean photon numbers and a Fokker-Plank-type equation for the mode-intensity distribution function are obtained. Approximate solutions of the steady-state master equation are examined.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

О кинетическом уравнении для двухмодового лазера

Фам Ле Киен, А.С.Шумовский

Исследованы динамика и статистика фотонов в двухмодовом лазере на основе ранее полученного кинетического уравнения. Построены замкнутые системы уравнений для средних чисел фотонов и уравнение типа Фоккера – Планка для функции распределения интенсивностей мод. Получены приближенные решения для равновесного случая и некоторые предсказания о статистических свойствах фотонов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

In the recent paper /1/, to be referred hereafter as I we derived the master equation for the photon distribution in a two-mode multiphoton laser. This equation was obtained from the exactly soluble model of a three-level plus two-mode system with multiphoton transitions /2/, using the Scully procedures /3/ and the one-atom approximation /4/. In general, we have proposed in I that pumping occurs for all three levels of the system.

In this paper we explore the above-mentioned equation in some particular cases and limits. We use the same symbols as in I, and therefore we do not repeat the two-mode multiphoton laser model. This model is described in detail in I .

At the beginning of our analysis we assume for simplicity that all atoms of the laser active medium are pumped only to the upper level 3. Then, the master equation obtained in I reads

$$\frac{d}{dt}p(n_1, n_2) = -p(n_1, n_2)\frac{A_1 G_1(n_1+m_1) + A_2 G_2(n_2+m_2)}{1+\frac{B_1}{A_1}G_1(n_1+m_1)+\frac{B_2}{A_2}G_2(n_2+m_2)} +$$

$$+ p(n_1-m_1, n_2)\frac{A_1 G_1(n_1)}{1+\frac{B_1}{A_1}G_1(n_1)+\frac{B_2}{A_2}G_2(n_2+m_2)} + \tag{1}$$

$$+ p(n_1, n_2-m_2)\frac{A_2 G_2(n_2)}{1+\frac{B_1}{A_1}G_1(n_1+m_1)+\frac{B_2}{A_2}G_2(n_2)} +$$

$$+ C_1(n_1+1)p(n_1+1, n_2) + C_2(n_2+1)p(n_1, n_2+1) -$$

$$- C_1 n_1 p(n_1, n_2) - C_2 n_2 p(n_1, n_2).$$

Here $p(n_1, n_2)$ is the distribution function of photons in the modes, A_α's and B_α's are gain and nonlinear coefficients, respectively, C_α's are loss parameters. The numbers m_α are the multiples of the transitions $3 \to \alpha$ sharing the common upper level 3. The functions $G_\alpha(n_\alpha)$ are

$$G_\alpha(n_\alpha) \equiv \frac{n_\alpha!}{(n_\alpha - m_\alpha)!}, \quad \alpha = 1, 2. \tag{2}$$

First of all we derive from eq.(1) the equations for the mean numbers of photons in the modes

$$\langle n_\alpha \rangle = \Sigma\, n_\alpha p(n_1, n_2), \quad \alpha = 1, 2. \tag{3}$$

They are

$$\frac{d\langle n_\alpha \rangle}{dt} = m_\alpha \Sigma \left(\frac{A_\alpha G_\alpha(n_\alpha + m_\alpha)}{1+\frac{B_1}{A_1}G_1(n_1+m_1)+\frac{B_2}{A_2}G_2(n_2+m_2)} - C_\alpha n_\alpha \right) p(n_1, n_2). \tag{4}$$

If the quantum correlations are small:

$$\langle n_1^k n_2^\ell \rangle \simeq \langle n_1 \rangle^k \langle n_2 \rangle^\ell , \tag{5}$$

equations (4) reduce to the following closed set of equations for the mean photon numbers:

$$\frac{d\langle n_\alpha \rangle}{dt} = m_\alpha \frac{A_\alpha G_\alpha (\langle n_\alpha \rangle + m_\alpha)}{1 + \frac{B_1}{A_1} G_1(\langle n_1 \rangle + m_1) + \frac{B_2}{A_2} G_2(\langle n_2 \rangle + m_2)} - C_\alpha \langle n_\alpha \rangle . \tag{6}$$

In order to describe the macroscopic behaviour of the laser we assume that $\langle n_\alpha \rangle \gg m_\alpha$. In this case equations (6) can be simplified as

$$\frac{d\langle n_\alpha \rangle}{dt} = m_\alpha \frac{A_\alpha \langle n_\alpha \rangle^{m_\alpha}}{1 + \frac{B_1}{A_1} \langle n_1 \rangle^{m_1} + \frac{B_2}{A_2} \langle n_2 \rangle^{m_2}} - C_\alpha \langle n_\alpha \rangle . \tag{7}$$

If we assume $(B_\alpha / A_\alpha) \langle n_\alpha \rangle^{m_\alpha} \ll 1$ we can get from eq. (7)

$$\frac{d\langle n_1 \rangle}{dt} = m_1 \left(A_1 \langle n_1 \rangle^{m_1} - B_1 \langle n_1 \rangle^{2m_1} - A_1 \frac{B_2}{A_2} \langle n_1 \rangle^{m_1} \langle n_2 \rangle^{m_2} \right) - C_1 \langle n_1 \rangle,$$

$$\frac{d\langle n_2 \rangle}{dt} = m_2 \left(A_2 \langle n_2 \rangle^{m_2} - B_2 \langle n_2 \rangle^{2m_2} - A_2 \frac{B_1}{A_1} \langle n_1 \rangle^{m_1} \langle n_2 \rangle^{m_2} \right) - C_2 \langle n_2 \rangle. \tag{8}$$

These equations have been obtained by expanding the denominator in eq. (7) and taking the terms up to $(B_\alpha / A_\alpha) \langle n_\alpha \rangle^{m_\alpha}$. They describe light amplification in a three-level medium with multiphoton transitions provided that correlations are neglected and the saturation effect does not begin to act , i.e.

$$m_\alpha \ll \langle n_\alpha(t) \rangle \ll (A_\alpha / B_\alpha)^{1/m_\alpha} . \tag{9}$$

In the case $m_1 = m_2 = 1$, $A_1 = A_2$ and $B_1 = B_2$ eqs. (8) are in strict agreement with the results of /5,6/. We now examine the steady-state photon statistics. For simplicity we consider the case of one-photon transitions $m_1 = m_2 = 1$. In the steady state the photon distribution function $p(n_1, n_2)$ is independent of time. The corresponding equation for this function is found from eq. (1) to be

$$\frac{A_1 n_1}{1+\frac{B_1}{A_1}n_1+\frac{B_2}{A_2}(n_2+1)}\, p(n_1-1, n_2) - C_1 n_1\, p(n_1, n_2) -$$

$$-\frac{A_1(n_1+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}\, p(n_1, n_2) + C_1(n_1+1)p(n_1+1, n_2) +$$

$$+\frac{A_2 n_2}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}n_2}\, p(n_1, n_2-1) - C_2 n_2\, p(n_1, n_2) - \tag{10}$$

$$-\frac{A_2(n_2+1)}{1+\frac{B_1}{A_1}(n_1+1)+\frac{B_2}{A_2}(n_2+1)}\, p(n_1, n_2) + C_2(n_2+1)p(n_1, n_2+1) = 0.$$

It is difficult to obtain the exact and explicit expression of the steady-state photon distribution from eq. (10). However, in the particular case of equal coupling parameters, i.e., when $A_1 = A_2 \equiv A$, $B_1 = B_2 \equiv B$ eq. (10) reduces to that derived by Singh and Zubairy in /6/. The steady-state photon distribution in this case has been found to be

$$p(n_1, n_2) = Z^{-1}\left(\frac{A^2}{BC_1}\right)^{n_1}\left(\frac{A^2}{BC_2}\right)^{n_2} / \Gamma\left(\frac{A}{B} + n_1 + n_2 + 2\right). \tag{11}$$

Here Z is the normalization constant. The dependence of the gamma function in the denominator of (11) on $(n_1 + n_2)$ is an evidence of the mode competition.

We proceed to approximating the solution of eq. (10) in some limit cases.

Let the terms $(B_a/A_a)n_a$, $(B_a/A_a)(n_a+1)$ in the denominators in eq. (10) be negligibly small compared with unity. Then, we can easily get the approximate solution

$$p(n_1, n_2) \simeq Z^{-1} \left(\frac{A_1}{C_1}\right)^{n_1} \left(\frac{A_2}{C_2}\right)^{n_2} , \tag{12}$$

which is a product of two Bose-Einstein distributions. The condition for validity of (12) is

$$A_\alpha < C_\alpha , \qquad \frac{C_\alpha - A_\alpha}{B_\alpha} \gg 1 . \tag{13}$$

Thus, when the laser action in each mode is much below threshold, the modes are statistically independent and chaotic. No mode competition is seen in this case.

We consider another case. Let us assume that $p(n_1, n_2)$ peaks at $(\hat{n}_1, \hat{n}_2)$ which are such that the terms $1 + (B_2/A_2)n_2$, $1 + (B_2/A_2) \times (n_2 + 1)$ in the denominators in (10) can be neglected. Then, eq. (10) can be simplified to read

$$\frac{A_1^2}{B_1} p(n_1 - 1, n_2) - C_1 n_1 p(n_1, n_2) - \frac{A_1^2}{B_1} p(n_1, n_2) +$$

$$+ C_1 (n_1 + 1) p(n_1 + 1, n_2) + \frac{A_1 A_2}{B_1 \hat{n}_1} n_2 p(n_1, n_2 - 1) - C_2 n_2 p(n_1, n_2) - \tag{14}$$

$$- \frac{A_1 A_2}{B_1 \hat{n}_1} (n_2 + 1) p(n_1, n_2) + C_2 (n_2 + 1) p(n_1, n_2 + 1) = 0 .$$

The solution of eq. (14) is easily found to be

$$p(n_1, n_2) = Z^{-1} [(A_1^2/B_1 C_1)^{n_1} / n_1!] \left(\frac{A_2/C_2}{A_1/C_1}\right)^{n_2} . \tag{15}$$

The conditions for validity of (15) are

$$A_1/C_1 \gg 1 ,$$

$$(A_1/C_1 - A_2/C_2) A_1/C_1 \gg B_2/C_2 . \tag{16}$$

Three conclusions, as seen from (15), can be made, if the parameters of the laser satisfy the conditions (16). The first conclusion is that the two modes tend to become statistically independent in spite of the occurrence of mode competition in the laser action. The second conclusion is that the distribution of photons in mode 1 is poissonian and independent of the action of the laser in mode 2. The last conclusion is that the distribution of photons in mode 2 is chaotic and strongly effected by mode 1 even at gain that is much above threshold. This fact is clearly a result of mode competition.

From (15) the mean numbers of photons in the modes are found to be

$$\langle n_1 \rangle_o = A_1^2/B_1 C_1, \quad \langle n_2 \rangle_o = \frac{A_2/C_2}{A_1/C_1 - A_2/C_2}. \tag{17}$$

According to I, the gain parameters A_1, A_2 and the nonlinear parameters B_1, B_2 are given by

$$A_\alpha = 2R(g_\alpha/\gamma)^2, \quad B_\alpha = 8R(g_\alpha/\gamma)^4, \quad \alpha = 1, 2. \tag{18}$$

Here R is the pump rate, γ is the decay constant and $g_\alpha (\alpha = 1, 2)$ are the atom-mode coupling parameters. From eqs. (17) and (18) we can see that if the pump rate is increased, the steady-state mean photon number of the mode 1 grows whereas the one of the modes 2 remains unchanged. We also note the interesting fact that if $g_1 > g_2$, the conditions (16) for validity of eq. (15) can be satisfied so that $\langle n_1 \rangle_o < \langle n_2 \rangle_o$ This means that in the laser with unequal coupling parameters the pump rate and the loss parameters can be chosen so that the steady state photon statistics of the mode with the larger mean photon number is chaotic whereas that of the mode with the smaller mean photon number is coherent.

Finally, in order to compare with earlier treatments, we convert eq. (10) into an equation which corresponds to the Fokker-Plank equation for the intensity distribution function. This can be done by using the representation of

$$p(n_1, n_2) = \int P(I_1, I_2)\, e^{-I_1 - I_2} \frac{I_1^{n_1} I_2^{n_2}}{n_1!\, n_2!} dI_1\, dI_2, \tag{19}$$

where I_1 and I_2 are the intensities, and $P(I_1, I_2)$ corresponds to the intensity distribution function. We can easily find that eq. (10) is equivalent to the following "Fokker-Plank-type" equation:

$$\Big\{ - A_1 \partial_{I_1} I_1 (1 - \partial_{I_1}) - A_2 \partial_{I_2} I_2 (1 - \partial_{I_2}) +$$

$$+\left(C_1 \partial_{I_1} I_1 + C_2 \partial_{I_2} I_2\right)\left[1 + \frac{B_1}{A_1} I_1 (1 - \partial_{I_1}) + \frac{B_2}{A_2} I_2 (1 - \partial_{I_2})\right]\Bigg\}W = 0. \qquad (20)$$

Here, the auxiliary function $W(I_1, I_2)$ has been introduced by

$$P(I_1, I_2) = \left[1 + \frac{B_1}{A_1} I_1 (1 - \partial_{I_1}) + \frac{B_2}{A_2} I_2 (1 - \partial_{I_2})\right] W(I_1, I_2). \qquad (21)$$

In the case when $A_1 = A_2$ and $B_1 = B_2$, equations (20) and (21) coincide with those obtained in /6/.

Thus, in this paper the two-mode laser master equation previously derived in I has been used to investigate the photon dynamics and statistics in the laser. The closed sets of equations for the mean photon numbers, the "Fokker-Plank-type" equation for the mode intensity distribution and a number of predictions about the photon statistics in the steady-state have been obtained.

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